An Example From Class...

- A total charge Q is uniformly spread out over a flat disk of radius R. What is the electric field @ a point directly above or below the center of the disk?

- Spreading charge over a disk gives a surface charge density, so we need to evaluate:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S da'\,\sigma(\vec{r}\,')\,\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

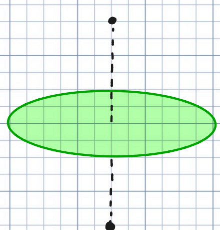
The point where we wish to know the electric field.

Visit every pt. on the disk...

... and add up the electric field produced @ $\vec{r}$ by the infinitesimal bit of charge $dq(\vec{r}\,') = da'\,\sigma(\vec{r}\,')$ found @ $\vec{r}\,'$.

- In this case the charge Q is spread uniformly over the surface, and the area of a disk is πR^2, so the surface charge density will be constant: $\sigma = Q/\pi R^2$.

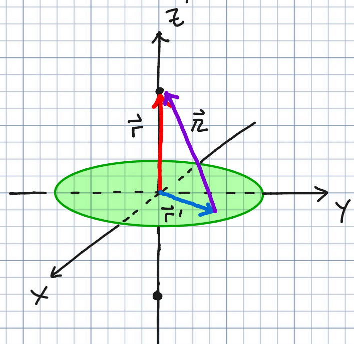
- First, how do we describe the disk and the point where we want to know $\vec{E}$?



Always try to describe things in the simplest manner possible. The description should take symmetry into account! In this case, points above & below the center lie on an axis of symmetry.

It makes sense to use Cylindrical Polar Coordinates here. The z-axis can be the axis of symmetry, and it's easy to describe the disk.

- We'll put the origin @ the center of the disk. Then:



Disk: $z = 0$, $0 \le s' \le R$, $0 \le \phi' < 2\pi$

Pt. on disk: $\vec{r}\,' = s'\hat{s} + 0\hat{z}$

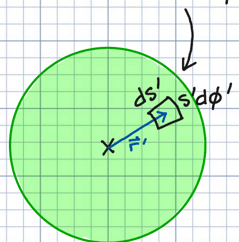
$$= s'\cos\phi'\hat{x} + s'\sin\phi'\hat{y} + 0\hat{z}$$

Pt. above/below center of disk: $\vec{r} = 0\hat{x} + 0\hat{y} + z\hat{z}$

Sep. vector: $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$$= -s'\cos\phi'\hat{x} - s'\sin\phi'\hat{y} + z\hat{z}$$

$$da' = s'd\phi'ds'$$



$$|\vec{\mathscr{r}}| = \sqrt{\underbrace{s'^2\cos^2\phi' + s'^2\sin^2\phi'}_{s'^2} + z^2}$$

$$= \sqrt{s'^2 + z^2}$$

<u>NOTE</u>: We used Cartesian unit vectors in $\vec{\mathscr{r}}$, since $\hat{s}$ depends on one of the coords we'll integrate over (ϕ').

- Before jumping right into the integral, work out the integrand:

$$da'\,\sigma(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = d\phi' ds' s' \frac{Q}{\pi R^2} \frac{(-s'\cos\phi'\hat{x} - s'\sin\phi'\hat{y} + z\hat{z})}{(s'^2 + z^2)^{3/2}}$$

$$= d\phi' ds' \frac{Q}{\pi R^2} \frac{(-s'^2\cos\phi'\hat{x} - s'^2\sin\phi'\hat{y} + s'z\hat{z})}{(s'^2 + z^2)^{3/2}}$$

- Putting this all together:

$$\vec{E}\underbrace{(0,0,z)}_{\text{Pt. above/below center of disk}} = \frac{1}{4\pi\varepsilon_0} \underbrace{\frac{Q}{\pi R^2}}_{\sigma \text{ is constant here}} \underbrace{\int_0^{2\pi} d\phi' \int_0^R ds'}_{\text{Visit every point}} \frac{(-s'^2\cos\phi'\hat{x} - s'^2\sin\phi'\hat{y} + s'z\hat{z})}{(s'^2 + z^2)^{3/2}}$$

- Now, we could just evaluate this. But it's worth asking what we *expect* the answer to be.

- Based on the symmetry of the charge distribution (it is spread evenly over the disk) and the point where we're evaluating $\vec{E}$, we expect to get an electric field pointing straight up or down along that symmetry axis. So $\vec{E}$ should have a z-component, but no x- or y-component.

- Can we see this in our integral? Look @ the x-component:

$$E_x(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_0^R ds' \int_0^{2\pi} d\phi' \frac{(-s'^2\cos\phi')}{(s'^2+z^2)^{3/2}}$$

The only ϕ'-dependence in the integrand is a factor of $\cos\phi'$, which gets integrated over a full period: $0 \to 2\pi$.

$$\int_0^{2\pi} d\phi' \cos\phi' = 0$$

$$\hookrightarrow E_x(0,0,z) = 0$$

- The same is true for E_y. So we have:

$$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{z}\int_0^{2\pi} d\phi' \int_0^R ds' \frac{s'z}{(s'^2+z^2)^{3/2}}$$

- There's no ϕ' dependence, so the integral over ϕ' just gives 2π:

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2} z\,\hat{z}\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}}$$

- To evaluate the integral over s', we make a change-of-variable:

$$u = s'^2 + z^2 \rightarrow du = 2s'ds' \rightarrow ds's' = \frac{1}{2}du$$

$$s'=0 \rightarrow u = z^2 \quad \& \quad s'=R \rightarrow u = R^2+z^2$$

$$\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}} = \int_{z^2}^{R^2+z^2} du \, \frac{1}{2} \frac{1}{u^{3/2}} = \frac{1}{2}\left(-\frac{2}{u^{1/2}}\right)\Big|_{z^2}^{R^2+z^2}$$

$$= -\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right)$$

$$= \frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{R^2+z^2}}$$

- We have to be careful here. Our integrand was explicitly positive: s' is b/t 0 & R, & $z^2 > 0$, so $s'/(s'^2+z^2)^{3/2} \geq 0$. And if we add up lots of non-negative numbers, the result must also be non-negative. But z could be positive (above the center) or negative (below the center), so we must write the result of the s' integral as

$$\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}} = \frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}$$

- So our final answer is:

$$\frac{z}{|z|} = \begin{cases} 1, & z>0 \\ -1, & z<0 \end{cases}$$

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$$

If $Q>0$, this points up for $z>0$ & down for $z<0$, as expected.